NETWORKED ESTIMATION UNDER CONTENTION-BASED MEDIUM ACCESS

MABEN RABI, LUCA STABELLINI, ALEXANDRE PROUTIERE, AND MIKAEL JOHANSSON

Abstract. This paper studies networked estimation over a communication channel shared by a contention-based medium access protocol. A collection of N identical and physically decoupled scalar systems are sampled without sensor noise and transmitted over a common channel, using a contention-based medium access mechanism. We first carry out a calculation of the average distortion in estimation with irregular samples. Given the rate of packet generation at sensors, we characterize the traffic characteristics of the some contention based MAC schemes. This lets us derive the statistics of inter-arrival times which in turn allows us to compute the packet loss rates and also the statistics of delay within a sample period. Using these results, we track the estimation performance as the sample generation rate and the number of contending nodes are varied. We provide a heuristic rule-of-thumb for choosing the sampling interval which minimizes the average distortion. By combining the network traffic characterization with that of the estimation performance, we show this rule performs pretty well. Carrying along the same lines, we are able to compute the scaling limits of estimation performance with respect to the number of contending nodes.

1. Introduction

Since the first application of wireless in industrial control almost a 100 years ago, the number of actual deployments have remained small [17]. For a long time, the market has been limited to specific target applications (e.g. wireless remote controls) engineered using customized technologies and sometimes even operating on licensed spectrum. There is a growing consensus that this trend is now about to change: the enormous success of short-range wireless in home and office applications has raised consumer confidence in wireless technologies; the emergence of standardized, low-cost, low-power radios has made industrial wireless economically attractive compared to cabled sensors [10]. Intense efforts on wireless sensor networks [11] and networked control [2] indicates that a large class of industrial process could be reliably controlled despite deficiencies of the wireless medium. All together, this raises expectations of a wide deployment of industrial wireless [13]. The trend is supported by major standardization bodies and automation system vendors working actively on several standards for industrial wireless, including Zigbee [3], 6loWPAN [14], wirelessHART [8], ISA SP-100 and Bluetooth ULP.

Maben Rabi and Mikael Johansson are at the Automatic Control Lab, School of Electrical Engineering, Royal Institute of Technology (KTH), Stockholm, Sweden. E-mail: `firstname.lastname@radio.kth.se`.

Luca Stabellini is at the Department of Communication Systems, School of Information and communication Technology, KTH, Kista, Sweden. E-mail: `firstname.lastname@radio.kth.se`.

Alexandre Proutiere is at Microsoft Research, 7 JJ Thomson Avenue, CB3 0FB, Cambridge, UK. E-mail: `firstname.lastname@microsoft.com`.

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A typical industrial process might have several thousands of sensors and actuators, and a wide adoption of wireless technologies could mean that several hundreds of these are candidates for cable replacement by wireless. Since the wireless medium is shared, there are natural limits on the number of control loops that can be accommodated. Such limits could either be theoretical (e.g. combining the Shannon capacity of the wireless channel [7] with a minimum bit-rate requirement for stabilization of a linear system [21, 22]) or practical (e.g. combining constraints on actual medium access control (MAC) mechanisms with performance objectives beyond stabilization [23, 1, 6]). The contributions of this paper are of the second kind.

Specifically, we consider the problem of state estimation where sensor measurements are sent over a medium shared using a contention-based access mechanism. We believe that this problem is particularly relevant, since state estimation is a key component of modern automation systems and contention-based medium access is supported in most standards (e.g. as contention access periods in Zigbee, or in shared time slots in wirelessHART). Clearly, the estimator performance depends on the dynamics of the individual systems and the nominal sampling interval. More interestingly, it also depends on the distribution of transmission delays and loss rates of sensor packets which, in turn, depend on the specific MAC scheme and the number of contending nodes. The key contribution of this paper is to analytically quantify these interdependencies. For analytical tractability, we assume that the dynamic systems whose states we want to track are scalar and have the same time constants, and derive an explicit formula of how the expected estimator performance depends on sampling frequency, packet loss rate and inter-arrival times of samples. Although the problem of state estimation under random sampling and loss have received significant attention (*e.g.* [9, 12, 19]), we are unaware of any similar results in the literature. We briefly comment on implications of our results on random sampling policies and highlight the importance of carrying out an adequate continuous-time analysis of the system performance. Admittedly, the time constants of individual systems vary in real deployments, but we demonstrate that homogeneous time constants is a worst-case (since, everything else being equal, the achievable performance improves if some time constants are smaller) which justifies our approach. To characterize packet loss rates and packet inter-arrivals, we focus on the case where the sensor measurements are taken at the same time instant, and analyze the MAC performance for systems with geometrically distributed (as in classic slotted-Aloha systems) or uniformly distributed (as suggested by IEEE802.15.4 standard) contention window. Also in this area, we are aware of very few results (*e.g.*, [16]). In addition, the analysis of queueing systems with transient and correlated traffic is substantially harder than for saturated sources. We combine these two contributions into an analytical model for how the estimator performance scales with the number of nodes. Extensive numerical examples highlight the analytical results.

The paper is organized as follows. Section 2 describes the problem formulation and general assumptions. Section 3 studies estimation under random delays and packet losses, and derives closed-form expression for the expected estimator performance. In Section 4, we carry out the traffic calculation which lead to the statistics of inter sample times and also of the packet loss rate. In Section 5, we combine the results of sections 3,4 and provide a heuristic rule for choosing the optimal sampling interval and provide justification for it. We also describe the scaling behaviour with respect to the number of competing nodes. In the final section, we summarize the conclusions of this work.

2. Problem formulation and assumptions

We consider N scalar plants, each given by the dynamics

$$dx_t^{(i)} = ax_t^{(i)}dt + dW_t^{(i)}, \qquad i = 1, \dots, N \tag{1}$$

where $W_t^{(i)}$ is a standard one dimensional Wiener process independent of $x_0^{(i)}$. For analytical tractability, we assume that the states can be measured exactly, i.e. without noise. We assume that sensor i samples the ith plant state at times $s_k^{(i)} = kh + \phi_k^{(i)}$, and consider two particular cases: *synchronized sensing*, where $\phi_k^{(i)} = 0$ for all k and all i, and *independent sensing* where $\phi_k^{(i)}$ are independent random variables uniformly distributed on the interval $[0, h)$.

The samples are transmitted over a shared communications channel to the corresponding estimator nodes, see Figure 1. The N transmitters contend for the channel using a contention-based medium access scheme. For sake of simplicity we consider a slotted system. After the sample is generated, the transmitter waits a random number of slots before its first transmission attempt: the waiting time is either geometrically distributed (as the classical slotted ALOHA) or uniformly distributed (similar to CSMA). The contention causes collisions which requires retransmissions and gives rise to a random delay between the sampling instants and the times when estimator nodes receive their data. If a sensor has not been able to deliver its data before a new sample is generated, it attempts to transmit the new data and discards the old one.

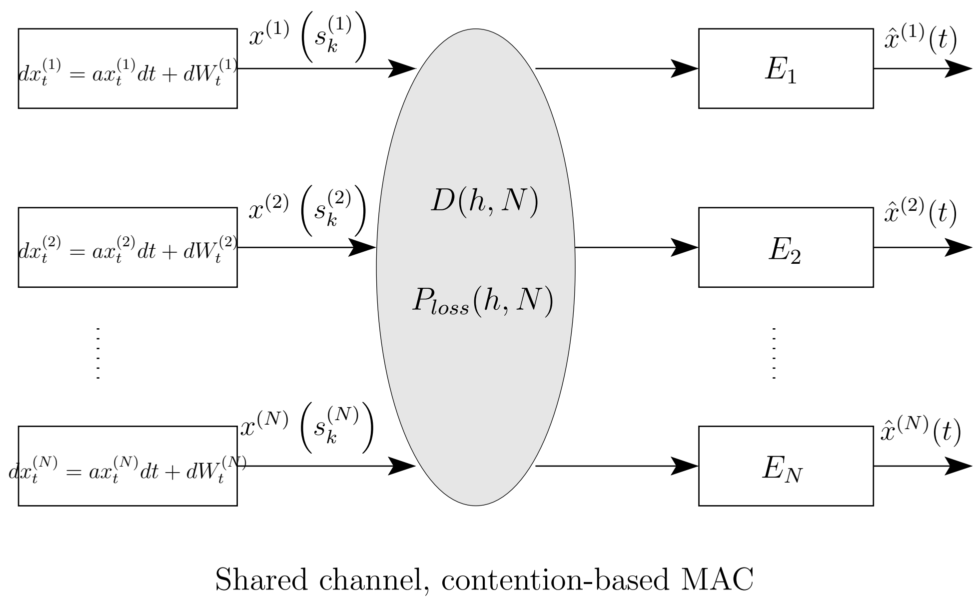

Figure 1. The estimation problem setup: the states of N identical plants are estimated via samples transmitted over a shared channel. Samples could be delayed and potentially lost because of contention.

We are interested in maintaining estimates of the process states so that the average distortion

$$J_e \triangleq \frac{1}{N}\sum_{i=1}^{N} \limsup_{M\to\infty} \frac{1}{M}\int_0^M \mathbb{E}\left[\left(x_t^{(i)} - \hat{x}_t^{(i)}\right)^2\right] dt \tag{2}$$

is minimized. The distortion depends on the process time constants a and the noise intensities, but also on the MAC delays and loss probabilities, and hence on the number of contending nodes. Our problem is to develop analytical model for these dependencies: how the distortion depends on the process time constants, average sampling rates, MAC delay and loss rate; how the delays and loss rates depends on sampling scheme, MAC protocol, sampling interval and number of contending nodes; and how the overall system performance can be made to scale with the number of contending nodes and the system time constants.

3. Estimation under random delays and losses

In this section, we study the expected performance for estimators operating under random delays and losses. We focus on scalar systems with noise-free observations and derive closed-form expressions for the mean-square distortion, first for the case of no contention delay, and later for the combined delay and loss scenario We give some insight related to optimal randomized sampling policies, highlight why it is important to consider a continuous-time analysis, and demonstrate that in this framework, our assumption that all systems have the same drift term corresponds to analyzing the worst-case scenario.

3.1. Analytical expressions for expected performance under time-varying sampling.

Consider a scalar continuous-time process that obeys the stochastic differential equation,

$$dx_t = ax_t dt + dW_t, \tag{3}$$

with W_t being a standard one dimensional Wiener process independent of x_0. The state process is assumed to be received with zero transmission delay at (possibly irregular but always causal) instants $q_0 < q_1 < \cdots < q_k < \cdots < \infty$. The q-sequence can be seen as a subsequence (to account of packet losses) of the s-sequence from the previous section. But the discussion below is valid for any causal sequence $\{q_k\}$ of reception times.

Assume that the process is estimated between the instants of successful transmissions by the least squares estimator:

$$\hat{x}_t = x_{l_t} e^{a(t-l_t)}. \tag{4}$$

Consider the mean-square distortion:

$$J_e \triangleq \limsup_{M\to\infty} \frac{1}{M} \int_0^M \mathbb{E}\left[(x_t - \hat{x}_t)^2\right] dt,$$

The error process is given by $e_t = x_t - \hat{x}_t$, which obeys:

$$de_t = dx_t - d\hat{x}_t = ae_t dt + dW_t.$$

Let $Y_t = e_t^2$, and use Itô's formula [15] to determine:

$$dY_t = de_t^2 = 2e_t de_t + \frac{1}{2}dt = (2aY_t + 1)\, dt + 2Y_t^{1/2} dW_t.$$

This equation is solved to be:

$$Y_t = \int_0^t (2aY_s + 1)\, ds + \int_0^t 2Y_s^{1/2} dW_s.$$

Take the expectation, and use the fact that the expected value of an Itô integal is zero, to get:

$$\mathbb{E}\left[Y_t\right] = \int_0^t (2a\mathbb{E}\left[Y_s\right] + 1)\, ds,$$

which can be rewritten as:

$$d\mathbb{E}\left[Y_t\right] = 2a\mathbb{E}\left[Y_s\right] dt + dt. \tag{5}$$

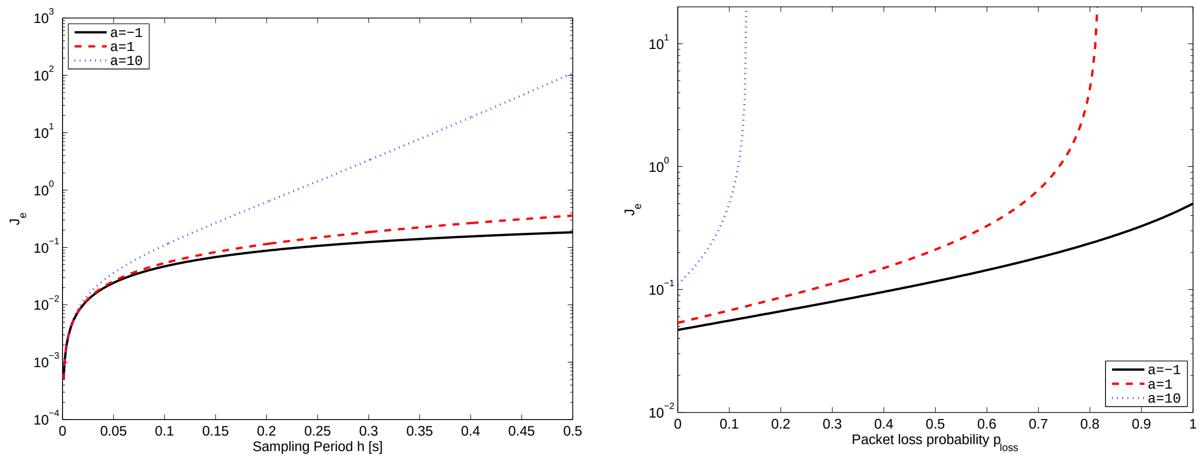

FIGURE 2. **Estimation performance under periodic sampling and and IID losses.**On the left, we plot the estimation distortion under periodic generation and reception of samples (no losses). Notice that the distortion is always finite regardless of the value of h. On the right, we plot the estimation distortion for a fixed h (sample generation period), but with varying loss rates. Notice that for unstable systems, finite distortion is possible only if the sample loss rate is small enough.

The solution to Equation (5) gives the expected value of the squared error process as:

$$\mathbb{E}\left[e_t^2\right] = \mathbb{E}\left[Y_t\right] = \frac{e^{2at}-1}{2a} - e^{2at}\mathbb{E}\left[e_0^2\right].$$

Because the expected error variance at time $t=0$ is zero, we have:

$$\mathbb{E}\left[e_0^2\right] = 0,$$

and hence:

$$\mathbb{E}\left[e_t^2\right] = \frac{e^{2at}-1}{2a}, \tag{6}$$

which gives the integral of the squared estimation error under an inter-sample interval is given by

$$\int_{q_{j-1}}^{q_j} \mathbb{E}\left[e_t^2\right] dt = \frac{e^{2a(q_j-q_{j-1})}-1}{4a^2} - \frac{q_j-q_{j-1}}{2a}. \tag{7}$$

Performance under Bernoulli packet losses. As an example of the above result, consider the situation where the variations in the sample reception times is caused by packet losses, and the underlying loss process is Bernoulli with loss probability p. Then,

$$\mathbb{P}\left[q_k - q_{j-1} = nh\right] = (1-p)p^{(n-1)},$$

so that

$$\mathbb{E}\left[q_j - q_{j-1}\right] = \sum_{n=1}^{\infty}(1-p)p^{n-1}nh = \frac{h}{1-p},$$

and,

$$J_{\mathrm{e}} = \frac{1}{\mathbb{E}[q_j - q_{j-1}]} \sum_{n=1}^{\infty} (1-p)p^{n-1} \left(\frac{e^{2anh}-1}{4a^2} - \frac{nh}{2a} \right).$$

Under the assumption:

$$2a < \frac{1}{h} \ln \left(\frac{1}{p} \right), \tag{8}$$

the series converges and we find

$$J_{\mathrm{e}} = \frac{1}{4a^2 h} \frac{(1-p)^2 e^{2ah}}{(1-pe^{2ah})} - \frac{1-p}{4a^2 h} - \frac{1}{2a}. \tag{9}$$

Combined with the expression for how the loss probability during the contention phase depends on MAC parameters and the number of contending nodes, the above expression will be an integral part of our study of networked estimation under contention-based medium access.

Random sampling policies and continuous vs discrete time analysis. The calculations in the previous section can also be useful for establishing more general results on how the estimation performance depends on the distribution of inter-sample times. Specifically, assume that the mean sampling interval $\mathbb{E}[q_j - q_{j-1}]$ is fixed and equal to $\overline{h}$. Then (7) gives that

$$J_e = \frac{1}{\overline{h}} \mathbb{E}\left[f_c(q_j - q_{j-1}) \right],$$

with,

$$f_c(\Delta_k) = \frac{e^{2a\Delta_k}}{4a^2} - \frac{\Delta_k}{2a}.$$

Since f_c is a convex function, Jensen's inequality implies that J_e increases with increasing variance in the inter-sample times. This formalizes the folklore that "jitter hurts" and establishes that the optimal sampling policy is regular sampling. As a consequence, for equal average sampling rate $\overline{h}$, Bernoulli sampling is better than Poisson sampling (see [12] for a thorough examination of Poisson sampling). This follows from the fact that variance of the Poisson process $\sigma_p^2 = \overline{h}$ is greater than the variance of the Bernoulli process $\sigma_b^2 = \overline{h}(1-p)$, where p is the success probability. We also note in passing that the situation is very different if we only consider the performance at the sampling instants. Specifically, let

$$J_e^{(d)} \triangleq \limsup_{M\to\infty} \frac{1}{M} \sum_{j=1}^{M} \mathbb{E}\left[\left(x_{q_k} - \hat{x}_{q_j} \right)^2 \right] = \mathbb{E}[f_d(q_j - q_{j-1})],$$

where, by (6),

$$f_d(\Delta_k) = \frac{e^{2a\Delta_k}-1}{2a}.$$

Then, since f_d is convex for $a > 0$ and concave for $a < 0$, $J_e^{(d)}$ increases with increasing variance of the sampling interval when $a > 0$, but decreases with increasing variance of the sampling interval for $a < 0$ (indicating that jitter would actually be beneficial, cf. [12]). Thus, neglecting intersample behavior can be misleading and our subsequent investigations focuses solely on the continuous-time estimator performance.

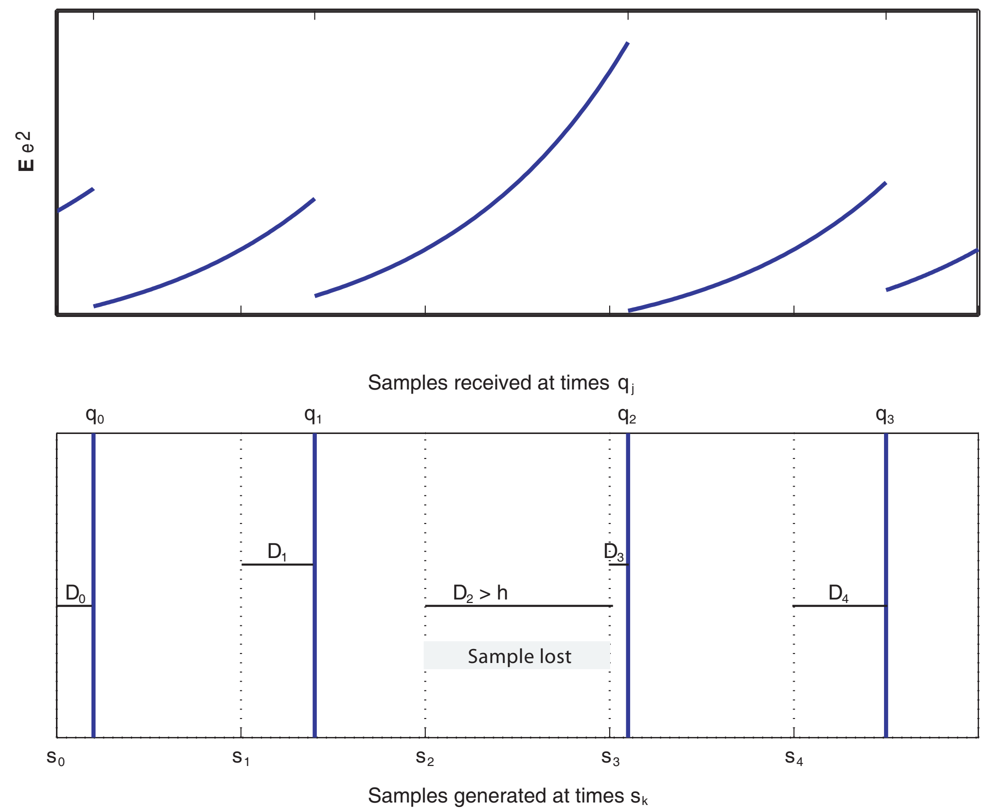

FIGURE 3. Timing of production and reception of samples: samples are generated at times s_k and subject to a contention-induced delay D_k. If a new sample is generated before the previous has been successfully transmitted, the old packet is discarded (resulting in a packet loss) and the more recent sample is transmitted. The reception times of samples are denoted q_j.

3.2. **Taking the MAC delay into account.** In this part, we compute the effect of the delay within a sampling period in addition to the effect of the sample losses. Note that a sample is declared lost if it could not be delivered within one sample period. Consider a sample generated at time instant $s_k = kh$. Assume that this sample has been successfully delivered within that sample period itself, but at a later time $kh + D_k$, where the MAC delay D_k is such that: $0 \leq D_k \leq h$; see Figure 3. Denote by $m-1$, the number of samples lost continuously henceforth. This means that the next successfully delivered sample is the sample generated at time $s_{k+m} = kh+mh$. Let the MAC delay for the delivery of this sample be D_{k+m}. By this convention, when a sample generated at time nh is not delivered within a following period, it simply means that $D_n > h$. Note also the the random sequence $\{D_n\}$ is an IID one. The sequence of reception times is denoted by $\{q_j\}$ where,

$$
\begin{aligned}
q_0 &= s_0 + D_0, \\
q_{j+1} &= \inf \{ s_k + D_k \,|\, D_k \leq h, s_k + D_k > q_j \}, \\
j &= \sum_{n=1}^{k} \mathbf{1}_{\{D_n \leq h\}} \qquad \leq k.
\end{aligned}
$$

Because the samples are noise-free values of the state at sampling instants, despite the delayed delivery, the corresponding least-squares estimator is still quite similar to the one in the equation (4).

$$\hat{x}_t = \begin{cases} \vdots & \vdots, \\ x_{\dots h} \times e^{a(t-\dots h)}, & \textbf{if } \dots h + D_{\dots} \le t < kh + D_k, \\ x_{kh} \times e^{a(t-kh)}, & \textbf{if } kh + D_k \le t < (k+m)h + D_{k+m}, \\ x_{(k+m)h} \times e^{a(t-(k+m)h)}, & \textbf{if } (k+m)h + D_{k+m} \le t < \dots h + D_{\dots}, \\ \vdots & \vdots. \end{cases} \tag{10}$$

This is because of the Markov property of the x-process. If $s, s_1, s_2, \dots, s_K$ are times such that

$$s > s_1 > s_2 > \dots > s_K,$$

then,

$$\mathbb{P}\left[x_t \in A \middle| x_{s_1}, x_{s_2}, \dots, x_{s_K}\right] = \mathbb{P}\left[x_t \in A \middle| x_{s_1}\right].$$

Since we incorporate the effect of the MAC delay, the recalculated mean-square distortion:

$$J_e \triangleq \limsup_{M\to\infty} \frac{1}{M}\int_0^M \mathbb{E}\left[(x_t - \hat{x}_t)^2\right] dt,$$

will be higher than before.

The error process is given by $e_t = x_t - \hat{x}_t$, which obeys:

$$de_t = dx_t - d\hat{x}_t = ae_t dt + dW_t$$

But with probability one, the estimation error variance is non-zero even at instants when a sample is delivered. The expected value of the squared error process is:

$$\mathbb{E}\left[e_t^2\right] = \frac{e^{2a(t-kh)} - 1}{2a}, \quad \forall \;\; kh + D_k \le t < (k+m)h + D_{k+m}.$$

Hence, the expected integral of the squared estimation error under an inter-sample interval is given by:

$$\begin{aligned} \int_{q_j}^{q_{j+1}} \mathbb{E}\left[e_t^2\right] dt &= \int_{kh+D_k}^{(k+m)h+D_{k+m}} \mathbb{E}\left[e_t^2\right] dt, \\ &= \frac{e^{2a[mh+D_{m+k}]} - 1}{4a^2} - \frac{e^{2aD_k} - 1}{4a^2} - \frac{mh}{2a}. \end{aligned} \tag{11}$$

Since the underlying loss process is Bernoulli with loss probability p, as before,

$$\begin{aligned} \mathbb{E}\left[q_{j+1} - q_j\right] &= \mathbb{E}\left[mh + D_{m+k} - D_k\right], \\ &= \mathbb{E}\left[mh\right], \\ &= \sum_{n=1}^{\infty} (1-p)p^{n-1} nh = \frac{h}{1-p}, \end{aligned}$$

and

$$J_{\mathrm{e}} = \frac{1}{\mathbb{E}\left[q_{j+1} - q_j\right]} \left\{ -\frac{\mathbb{E}\left[e^{2aD_k}\right] - 1}{4a^2} + \sum_{n=1}^{\infty} (1-p)p^{n-1} \left(\frac{e^{2a[nh+D_{n+k}]} - 1}{4a^2} - \frac{nh}{2a} \right) \right\}.$$

Now, under the same stability condition as before:

$$2a < \frac{1}{h} \ln \left(\frac{1}{p}\right),$$

the series converges and we find

$$J_{\mathrm{e}} = \mathbb{E}\left[e^{2aD_k}\right] \frac{(1-p)\left(e^{2ah}-1\right)}{4a^2h\left(1-pe^{2ah}\right)} - \frac{1}{2a}. \tag{12}$$

The quantity $\mathbb{E}\left[e^{2aD_k}\right]$ in the above expression needs to be evaluated. There are three means available to us. We could compute it exactly by numerical computation using the delay distribution calculated in section [citation comes here]. We could also evaluate it approximately by fitting a simple parametric distribution such as the exponential one for the delay statistics. Another route of approximation would be to assume the worst case MAC delay of h seconds.

3.3. **Assuming the same coefficient a leads to the worst case.** Here, we will see that the assumption of the same value for the drift coefficient (a) for the N different plants is a worst case assumption. Suppose that the a-values were different for the different plants, then the average estimation distortion (12) for the plants is a monotonic function of a with the result that the plant with the largest a value will be estimated with the largest average distortion. Hence, all other parameters being fixed, assuming that all systems have the largest possible value of a leads to a systems with the largest sum of aggregate distortions. Similarly, assigning the smallest values of a as the common value leads to the smallest sum of aggregate distortions. However when the a values are all different, the optimal sampling intervals could be all different. In sections 4, 5 we will see that the optimal sampling interval h^* is somewhat insensitive to the value of a. The practical solution would be compute the optimal sampling interval for the largest and median values of the a-coefficient.

The aggregate error (J_e) is a monotonically increasing function of a and will wee see this now. For any finite realization (κ_1, κ_2) of the pair (q_j, q_{j+1}), the integral

$$\int_{\kappa_1}^{\kappa_2} \frac{e^{2at}-1}{2a} dt,$$

is a differentiable and increasing function of Clearly, the above integral is a monotonically increasing function of a. To see this, notice that,

$$\frac{d}{da}\frac{e^{2at}-1}{2a} = e^{2at}\frac{2at-1+e^{-2at}}{4a^2t^2},$$

which is non-negative because the function $x-1+e^{-x}$ is non-negative. To verify this last claim, notice that:

$$e^{-x} \geq 1-x, \ \forall x \in \mathbb{R},$$

because the straight line defined by $y = 1-x$ is a tangent to the convex function e^{-x}.

Hence whenever the expectations of the integral is finite in an interval of a, it is also non-decreasing function of a in that interval.

4. Delay analysis in networks with independent or correlated traffic

As illustrated in the previous section, the estimation performance critically depends on the sample delays, as well as of the sample loss rate. In this section, we aim at characterizing the distribution of these delays and at evaluating the sample loss rate in a network of N interfering sensors. For simplicity, we assume that sensors directly transmit samples to estimator nodes; in other words, we consider a single-hop network, although we expect similar results in the case of multi-hop networks.
Traffic scenarios. We investigate the sample delays in two extreme relevant traffic scenarios. First we analyze the case where the interfering sensors generate estimates simultaneously, i.e., they are synchronized. Then, we consider the case where estimates at the various sensors are generated independently. In both scenarios, at a given sensor i, estimates are assumed to be generated according to a renewal process: if s_l^i denotes the time at which the l-th sample is generated, then the random variables $(s_{l+1}^i - s_l^i)$, $l = 0, 1, \ldots$, are IID with distribution $\mathcal{S}^i$. An important example is obtained when samples are periodically generated every h seconds; in this case, we have simply $\mathcal{S}^i = \delta_h$. Note also that in the case of synchronized sensors, we have for any sensor i, and any sample l, $s_l^i = s_l$, and hence $\mathcal{S}^i = \mathcal{S}$. We assume that if a sensor has not been able to transmit successfully a sample before a new sample is generated, it then tries to transmit the new estimate and discards the previous one.
MAC protocols. The sample delays are due to the fact that the sensors compete to access a common radio channel using some adaptive or non-adaptive MAC protocols. The objective of adaptive MAC protocols is usually to let each sensor continuously learn about the number of active interfering sensors. Often in sensor networks, the number of sensors is not evolving and actually, this number might be even known. This justifies why in the analysis we mainly focus on non-adaptive protocols. The results can be generalized to adaptive protocols as those used in IEEE802.15.4 systems. Unless otherwise specified, sensors use a non-adaptive CSMA protocol: time is slotted and the slot duration is denoted by L; when a sensor has a new sample to transmit, a contention window is drawn randomly according to some distribution $\mathcal{C}$ with mean $1/q_{tr}$ slots. The contention window is decremented after each empty slot, and the sensor may start transmitting only when it reaches zero. If two sensors attempt to transmit samples at the same time, they suffer from a collision and the samples have to be retransmitted. After a collision, a new contention window is drawn. Of particular interests are the cases where the contention window distribution $\mathcal{C}$ is the uniform distribution on $[0, 2/q_{tr}]$ (as suggested in 802.15.4 standards), and where $\mathcal{C}$ is a geometric distribution. In the latter case, a sensor starts transmitting at the beginning of each empty slot with probability q_{tr} as in classical slotted-Aloha. To facilitate the analysis, and unless otherwise specified, we assume that the contention window is geometrically distributed. Finally the transmission of a sample requires a single time-slot.

In the following we provide an analytical characterization of the processes representing the receptions of the samples generated by sensor i by the corresponding estimator node. Denote by D_l^i the delay of the l-th sample generated by sensor i, keeping in mind that if D_l^i is such that $s_l^i + D_l^i > s_{l+1}^i$, then the sample is discarded. We restrict our attention to symmetric systems where sensors have similar sample generation processes, and hence we may drop superscript i and denote by D_l the generic delay of the l-th sample.

4.1. **Synchronized sensors.**

In such a scenario, the system evolution can be represented by a renewal process with renewal epochs coinciding with those of sample generations. We then define by D the typical delay of a sample generated by a given sensor. Let us evaluate the distribution of D.

Let r_j be the random delay expressed in slots required for exactly j sensors to successfully transmit their samples. Then $r_j = \sum_{i=1}^{j} v_i$ where v_i is the duration in slots of the interval of time between the successful transmissions of the samples of the $(i-1)$-th and i-th sensors. We have: for all $k \geq 1$,

$$\mathbb{P}[v_i = k] = a_i(1-a_i)^{k-1}, \text{ where } a_i = (N-i+1)q_{tr}(1-q_{tr})^{N-i}. \tag{13}$$

The generating function of v_i is then defined by:

$$\phi_{v_i}(z) = \frac{a_i z}{1-(1-a_i)z},$$

and that of r_j is:

$$\begin{aligned}\phi_{r_j}(z) &= z^j \prod_{i=1}^{j} \frac{a_i}{1-(1-a_i)z} \\ &= z^j \sum_{i=1}^{j} \frac{b_i}{z-\frac{1}{1-a_i}},\end{aligned}$$

with

$$b_i \left[\left(z - \frac{1}{1-a_i} \right) \cdot \phi_{r_j}(z) z^{-j} \right]_{z=\frac{1}{1-a_i}}.$$

Hence using similar arguments as in [20], we deduce the distribution of r_j:

$$\mathbb{P}\left[r_j = k\right] = \begin{cases} 0 & \text{if } k < j, \\ \prod_{i=1}^{j} a_i \sum_{x=1}^{j} \frac{(a_x-1)^j}{\prod_{h=1,h\neq x}^{j}(a_x-a_h)}(1-a_x)^{k-j}, & \text{otherwise.} \end{cases} \tag{14}$$

By symmetry, we know that: for all $k \geq 0$, $\mathbb{P}\left[D = kL\right] = \frac{1}{N}\sum_{j=1}^{N} \mathbb{P}[r_j = k]$, and hence,

$$\mathbb{P}[D = kL] = \frac{1}{N} \sum_{j=1}^{N} \prod_{i=1}^{j} a_i \sum_{x=1}^{j} \frac{(a_x-1)^j}{\prod_{h=1,h\neq x}^{j}(a_x-a_h)}(1-a_x)^{k-j} 1_{k\geq j}. \tag{15}$$

From the distribution of D we can derive other quantities of interest. For example, the sample loss rate p is equal to:

$$p = \int_0^{\infty} \mathcal{S}(du) \sum_{k:kL>u} \mathbb{P}[D = kL]. \tag{16}$$

The above formula is obtained conditioning on the duration of the interval between the generations of two successive samples (remember that $\mathcal{S}$ is the distribution of this duration). Given that the next sample is generated after the previous one and a delay u, the sample is lost if and only if $D > u$. When the sensors generate samples periodically once every h seconds, the sample loss rate becomes:

$$p = \mathbb{P}\left[D > h\right] = \sum_{k \geq \lfloor h/L \rfloor} \mathbb{P}\left[D = kL\right]. \tag{17}$$

We may also derive the distribution of a typical sample inter-arrival time τ at a estimator node. In the case of periodic sampling with period h, we can write: $\tau = D_{l+M} - D_l + Mh$, where $M-1$ is the number of samples lost between the two received samples, D_l and D_{l+M} are the delays of the first and second received samples respectively. Note that the random variables D_l, D_{l+M} and M are independent, and then the distribution of τ is obtained by simple convolutions. In other words, it is characterized by:

$$\mathbb{P}[\tau = t] = \sum_{n=1}^{\infty} \sum_{k_1,k_2} 1_{\{nh+k_1L+k_2L=t\}} p^{n-1} \mathbb{P}[D = k_1 L]\mathbb{P}[D = k_2 L]. \tag{18}$$

4.1.1. *Heuristic choice for q_{tr}.* We may want to tune the value of the transmission probability q_{tr} so as to minimize the sample loss rate. In [18], it has been shown that in the case of periodic sampling, a good approximation for the optimal q_{tr} was given by:

$$q_{tr} = \frac{2}{N+2}.$$

This choice can be explained as follows. The sample loss rate can be evaluated according to the formula:

$$p(h) = \sum_{k=0}^{N} \frac{N-k}{N} \pi_k,$$

where, π_k is the probability that exactly k packets were successfully transmitted within one sample period. Using the expression for π_k derived in [18], we may claim that the loss rate is minimized when

$$\sum_{i=1}^{N} (N-i) \frac{1-a_{N-i+1}}{a_{N-i+1}}.$$

is minimized. Now the choice for q_{tr} is obtained approximating the term $N-i$ by $N-i+1$. With this approximation, the optimal q_{tr} is obtained minimizing:

$$\begin{aligned} &\sum_{i=1}^{N} (N-i+1) \frac{1-a_i}{a_i} \\ &= \sum_{i=1}^{N} (N-i+1) \frac{1-(N-i)\, q_{tr}(1-q_{tr})^{N-i}}{(N-i+1)\, q_{tr}(1-q_{tr})^{N-i}}, \end{aligned}$$

which is equivalent to minimizing:

$$\begin{aligned} &\sum_{i=1}^{N} (N-i+1) \frac{1}{(N-i+1)\, q_{tr}(1-q_{tr})^{N-i}}, \\ &= \sum_{i=1}^{N} \frac{1}{q_{tr}(1-q_{tr})^{N-i}}, \\ &= \frac{1-(1-q_{tr})^{N+1}}{q_{tr}^2(1-q_{tr})^{N}}. \end{aligned}$$

We then obtain $q_{tr} = \frac{2}{N+2}$. For all $N \geq 2$, the rule we have derived respects a lower bound for q_{tr}, namely $\frac{1}{N}$. This lower bound is is the optimal choice for probability of attempting transmissions when the system is saturated and all nodes are in contention for the channel in every slot.

4.2. **Independent sensors.** Let us now evaluate the delays in the case of independent sensors. In general, an exact analysis is not possible, because of the inherent correlations between the transmission attempts of the various sensors. To circumvent this difficulty, we use the heuristic developed by Bianchi [4] to quantify the performance of the IEEE802.11 DCF in saturated Wireless LANs. The heuristics consists in analyzing the transmissions of a single sensor assuming that the other sensors create a constant random interference. This approach can be generalized to the case of un-saturated nodes (which is usually more realistic in sensor networks), and it has been recently theoretically justified [5].

Denote by λ the stationary probability that a given sensor has a sample to transmit. Then the probability c that a sensor experiences a collision when attempting to use the wireless channel can be approximated by:

$$c = 1 - (1 - \lambda q_{tr})^{N-1}. \tag{19}$$

As previously, when isolating a given sensor, the system evolution can be represented as a renewal process with renewal epochs coinciding with those of sample generations. Let us consider that at time 0, a sample is generated, and let us fix the epoch u at which the next sample is generated. Define $s = \lfloor u/L \rfloor$. Consider a time-slot before the successful transmission of the sample created at time 0. The probability that the sample is not transmitted in this slot is $\alpha = (1 - q_{tr}) + q_{tr} c$. Now for all $k = 1, \dots, s$, the probability that the sample generated at time 0 is transmitted successfully at time-slot k is:

$$\mathbb{P}[D = kL] = q_{tr}(1 - c) \times \alpha^{k-1}.$$

Similarly:

$$\mathbb{P}[D > sL] = \alpha^s.$$

Note that $D > sL$ means that the sample is not transmitted before the generation of a new sample. We deduce the average proportion of time where the sensor has a sample to transmit in the interval $[0, u]$:

$$\frac{1}{u}\left(q_{tr}(1 - c)\sum_{k=1}^{s} k\alpha^{k-1} + u\alpha^s\right). \tag{20}$$

Finally applying the renewal theorem and averaging over all possible values of u, we get:

$$\lambda = \frac{\int_0^\infty \mathcal{S}(du)\left(q_{tr}(1 - c)\sum_{k=1}^{s} k\alpha^{k-1} + u\alpha^s\right)}{\int_0^\infty \mathcal{S}(du)u}. \tag{21}$$

When sampling is performed periodically once every h seconds, the stationary probability that a sensor has a sample to transmit simplifies to:

$$\lambda = \frac{q_{tr}(1 - c)}{h}\sum_{k=1}^{\lfloor h/L \rfloor} k\alpha^{k-1} + h\alpha^{\lfloor h/L \rfloor}. \tag{22}$$

Solving the system of equations (19)-(21), we get c and λ and the delay distribution. From there, as previously, we can deduce the sample loss rate and the inter-arrival of samples at the estimator nodes.

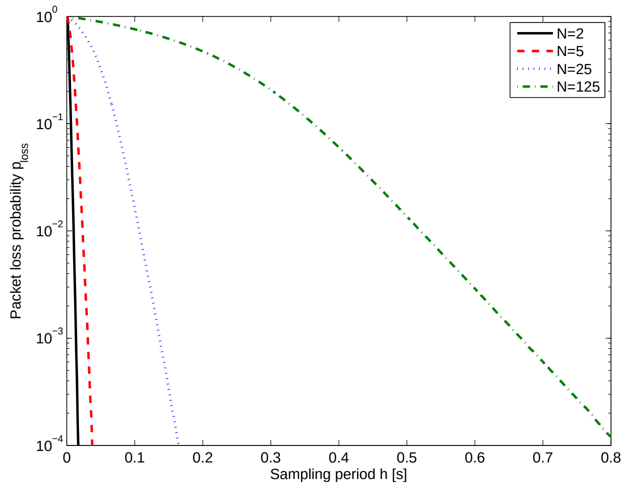

FIGURE 4. Sample loss rate in a fully synchronized system and periodic sampling.

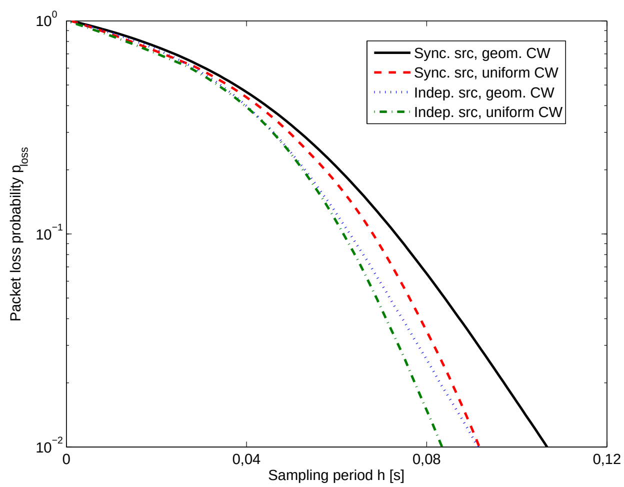

FIGURE 5. Sample loss rate with fully synchronized or independent sensors and geometric or uniform contention windows.

4.3. **Numerical experiments.** We conclude this section by presenting numerical experiments to illustrate the results obtained above. We consider periodic sampling only. All the results are obtained using the optimal heuristic choice for q_{tr}. The accuracy of this heuristic is illustrated in the last figure of the section. Unless otherwise specified, the distribution of the contention window is geometric.

In Figure 4, we evaluate the sample loss rate in the case of a fully synchronized system as a function of the sampling period h for various numbers of sensors N. Figure 5 is the analog of Figure 4, but here we add the case of independent sensors, and also compare the performance when choosing geometrically or uniformly distributed contention windows. The difference is not negligible, and uniform contention windows yield smaller loss rates.

Finally, in Figure 6, we compare our heuristic optimal transmission probability to the actual optimal probability obtained using numerical optimization. As shown, the heuristic is very accurate.

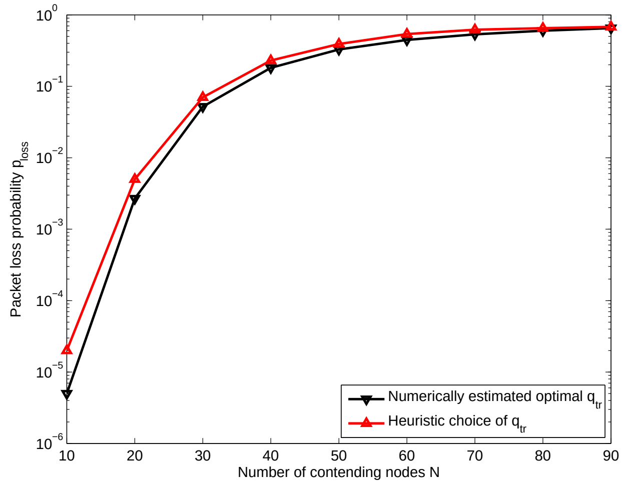

FIGURE 6. Accuracy of the heuristic choice for q_{tr}.

5. Networked estimation under contention-based medium access

As demonstrated in Section 3, real-time estimation based on a stream of samples has a quality dependent on the statistics of the inter-sample intervals and, to a lesser extent, also on the statistics of the MAC delay (since the delay is upper bounded by h). In section 4 we derived analytical models for how these parameters depend on the MAC scheme, the number of contending nodes and the rate at which new samples are generated. Our aim now is to study the achievable performance of estimation under contention-based medium access and characterize how this depends on critical system parameters.

To develop a basic intuition for the compound problem, consider the case of IID losses and zero MAC delay. The mean packet loss rate then completely determines the statistics of the interarrival times for samples. As illustrated in Figure 2(left) in Section 3, the estimation performance improves with decreasing sampling interval. On the other hand, as shown in Section 4, as the sampling interval decreases, fewer sensors succeed in transmitting their samples before the next one is generated, which results in an increased packet loss rate (see Figures 4 and 5) and a rapid deterioration in the estimation performance (Figure 2, right). Thus, when the sampling interval is small, the rate of samples being generated is high but so is the packet loss rate p. Increasing h lowers p but also decreases the rate of sample generation. Consequently, there should be an optimal sampling rate which balances the benefit of generating samples at a high rate with the deterioration caused by increased contention-induced MAC delays and loss rates. This behavior is clearly seen in Figure 5, which shows the estimator performance as function of sampling interval for a stable and an unstable system, respectively. In the stable case, the performance goes towards the steady state variance ($1/2a = 1/2$ in this case) as $h \to 0$ while in the unstable case, the distortion grows unbounded when the sampling rate is so high that the packet loss rate approaches the stability bound (8).

In the case of synchronized transmissions with a geometric contention window, we can draw an useful guideline. Although the distribution of inter-sample times is not exponential, roughly speaking, its higher moments are lowered when its mean is lowered. A glimpse of this property can be discerned in figure 8 . This property suggests a rule-of-thumb for picking a suitable sampling interval, namely the h that minimizes $\mathbb{E}\left[q_{j+1} - q_j\right]$. As observed in Figure 8, the

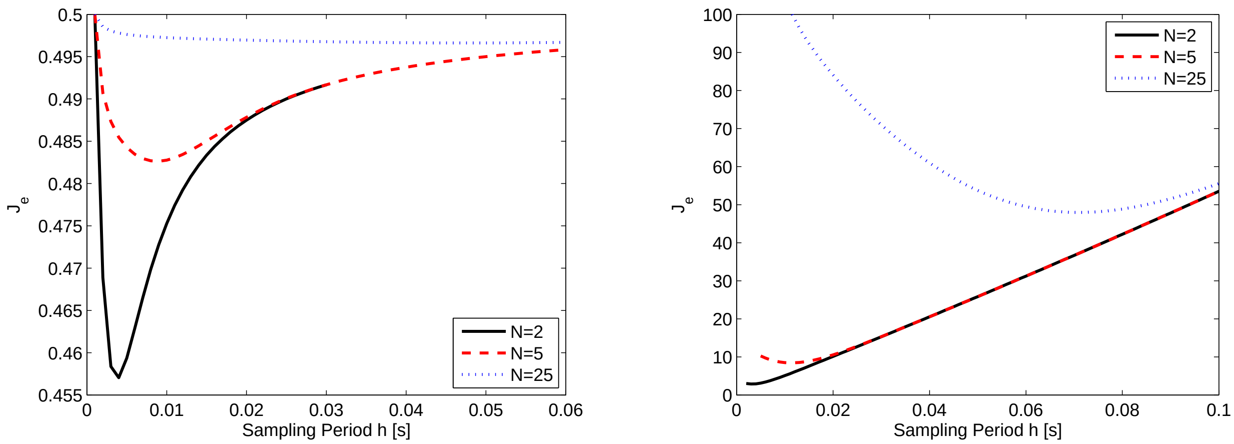

FIGURE 7. Estimator performance as function of sampling interval for a stable system with $a = -1$ (left) and an unstable system with $a = 0.001$ (right). In both cases, the number of contending nodes $N = 25$. The performance plots display a clear minimum which balances the generation of new samples with the contention-induced loss rate.

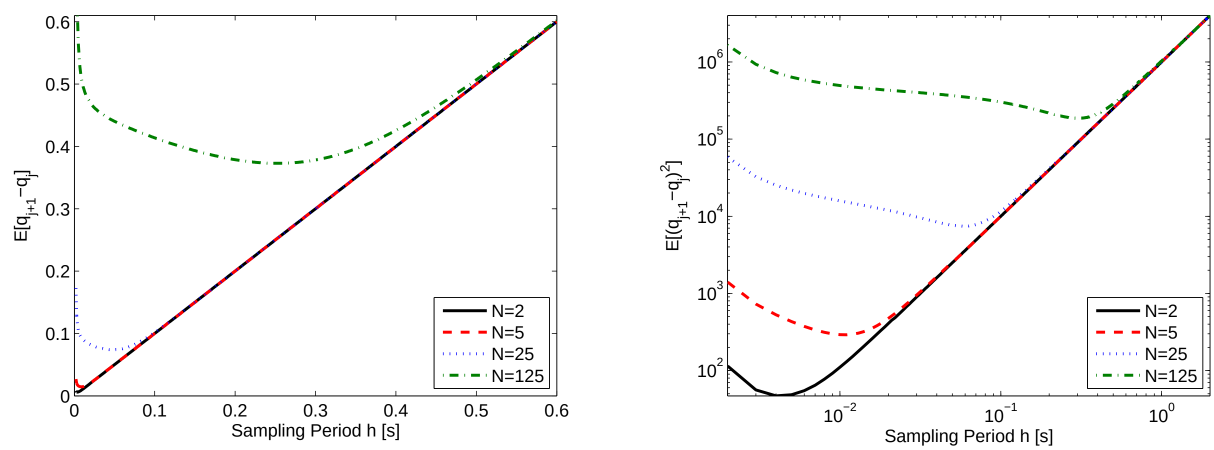

FIGURE 8. Variation of the first and second moments of the time between sample receptions as the sample generation period h changes.

sampling period $h^\star$ minimizing the average sample inter-arrivals, and thus leading to the best estimation performance, grows linearly with the number N of sensors. This observation seems valid both in the cases of independent and synchronized sensors. For independent sensors, we may justify the observation as follows. We know that the transmission probability q_{tr} has to scale as $1/N$. This implies that $\alpha = (1 - q_{tr}) + q_{tr}c$ roughly behaves as $1 - g/N$ for some positive constant g. The sample loss rate $p = \alpha^{\lfloor h/L \rfloor}$ then scales as $\exp(-g'h/N)$ where g' is another positive constant. Finally, the average sample inter-arrivals can be approximated by $h/(1-p)$, and one can easily show that the latter quantity is minimized for a value $h^\star$ of h that grows linearly with N. Justifying the linear growth in the case of synchronized sensors is more intricate, but the numerical experiments clearly illustrate this growth.

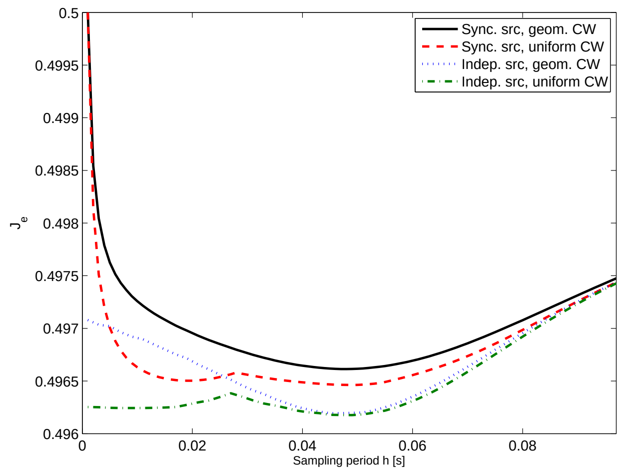

FIGURE 9. Estimation distortion for the four cases we considered. The curves have been generated assuming $N = 25$ nodes and $a = -1$.

Now, comparing this rule-of-thumb with the optimal sampling time selection shown in Figure 9,10 reveals that it works quite well also when we target the estimation distortion. For instance considering when there are 25 synchronized sources with stable plants, with uniform/geometric CW, the estimation distortion is minimized when $h = 0.051$ seconds. Instead, if we consider an unstable plant, the h minimizing the estimation distortion is 0.07 seconds. The value of h minimizing the mean inter-arrival time is $h = 2NL = 0.05$ seconds. However, the rule does not provide the true optimum.

When comparing the performance of independent and synchronized sensor transmissions, as described in Figures 9 and 10, the delays and the sample loss rate are usually better when sensor transmissions are independent. This echoes a popular theorem in traffic/queueing analysis that says that having batch arrivals (synchronized sensors) leads to a worse performance than that obtained when traffic is not generated in batches. However, while the performance metrics (delays and loss rate) are different for different transmission schemes, they are always of the same order. This explains also why we have a linear growth of $h^\star$ even in the case of synchronized sensors.

In Figures 9 and 10, we also notice a double-chin in the performance plots for transmission schemes with uniform contention windows. This is perhaps due to the sampling interval being close to the contention window.

We finally make a remark on scalability issues: once we fix the sampling period h, the packet loss rate p will monotonically increase with the number of contending nodes N. Depending on the specific value of a this will lead to a critical threshold which basically gives the maximum number of contending sensors which can be included in the system to maintain a bounded J_e (see equation (8)). As an example, we reported in figure 11 how the packet loss rate varies with the number of nodes assuming a sampling period $h = 0.1$ seconds, with fully synchronized sources and geometrically distributed back-off counters. The three horizontal lines denote the maximum acceptable loss rate to ensure stable estimation. Increasing the level of instability of the system decreases the maximum allowable number of nodes; in the same way, for a fixed a, increasing the sampling period will reduce the loss rate and thus allows to accommodate more sensors in the system.

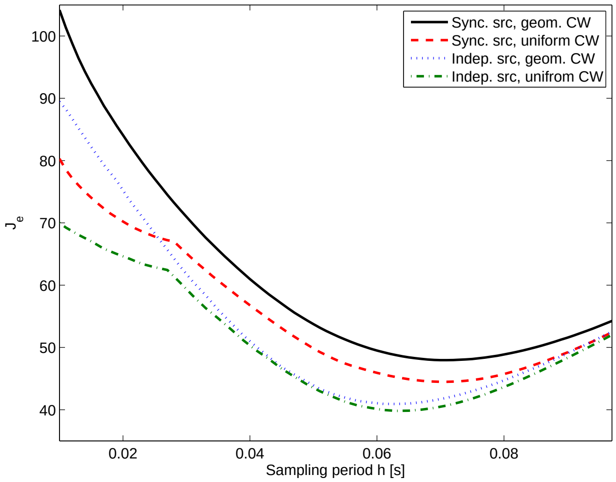

FIGURE 10. Estimation distortion for the four cases we considered. The curves have been generated assuming $N = 25$ nodes and $a = 0.001$.

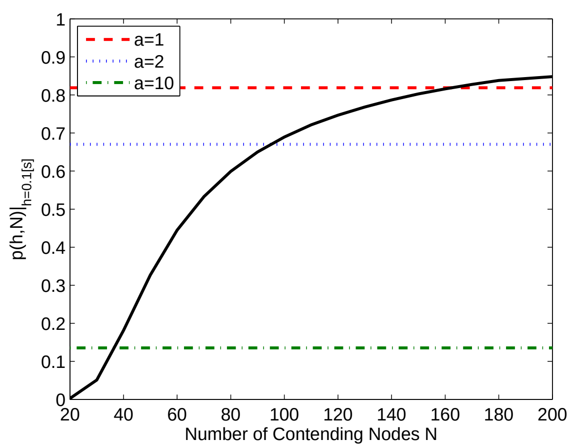

FIGURE 11. Packet loss rate for $h = 0.1[s]$ as a function of N. The three horizontal curves identify the maximum values of p which result in a bounded estimation.

Analogous considerations apply if we have a constraint on the maximum acceptable estimation distortion: increasing the number of nodes will also result in increased J_e as shown in Figure 12.

6. Conclusions

We have considered the problem of networked estimation over a communication channel shared by a contention-based medium access protocol. For analytical tractability, we have studied the situation when N identical scalar systems are sampled without sensor noise and transmitted over the channel, and focused on contention-based medium access mechanisms with geometric (ALOHA-like) and uniform (CSMA-like) contention windows. This has allowed us to derive closed-form expressions for how the expected estimator performance depends on the system dynamics, sampling interval, MAC delay and packet loss probability. The calculations give

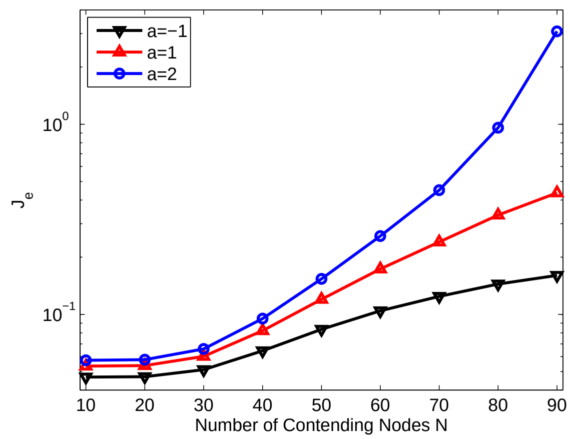

FIGURE 12. Estimation distortion as a function of N for $a = -1, 1, 2$.

insight into optimal randomized sampling policies and establishes the importance of considering a continuous-time performance criterion. We have also derived analytical models for the delay and loss probability distributions for MAC protocols with geometric and uniform contention windows under both synchronized and independent sensing times. For the case of geometric contention window, we derive a heuristic for optimal selection of the transmission probabilities, and discuss the optimal sampling time selection from a communications perspective. Integrating the two models allows us to study the compound problem, deriving guidelines for sampling time selection, and studying how the system performance scales with the number of sensor nodes and the degree of instability of the individual plants.

There are many open issues in our work. On the estimation side, it would be interesting to extend the work to cover noisy observations and (at least classes of) vector-valued systems. On the networking side, it would be useful to develop improved tools for studying networks with transient and correlated traffic, as well as short buffers where delayed packets must be discarded. It would also be interesting study systems with adaptive back-off counters such as 802.15.4, and develop improved and control-relevant MAC protocols.

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